

Standard Model Lagrangian (including neutrino mass terms)

From *An Introduction to the Standard Model of Particle Physics, 2nd Edition*,

W. N. Cottingham and D. A. Greenwood, Cambridge University Press, Cambridge, 2007,

Extracted by J.A. Shifflett, 5 Oct 2010.

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}\text{tr}(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}) - \frac{1}{2}\text{tr}(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}) && \text{(U(1), SU(2) and SU(3) gauge terms)} \\
& +(\bar{\nu}_L, \bar{e}_L)\tilde{\sigma}^\mu iD_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{e}_R\sigma^\mu iD_\mu e_R + \bar{\nu}_R\sigma^\mu iD_\mu \nu_R && \text{(lepton dynamical term)} \\
& -\frac{\sqrt{2}}{v} \left[(\bar{\nu}_L, \bar{e}_L)\phi M^e e_R + \bar{e}_R M^{e*}\bar{\phi} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \right] && \text{(electron, muon, tauon mass term)} \\
& -\frac{\sqrt{2}}{v} \left[(-\bar{e}_L, \bar{\nu}_L)\phi^* M^\nu \nu_R + \bar{\nu}_R M^{\nu*}\phi^T \begin{pmatrix} -e_L \\ \nu_L \end{pmatrix} \right] && \text{(neutrino mass term)} \\
& +(\bar{u}_L, \bar{d}_L)\tilde{\sigma}^\mu iD_\mu \begin{pmatrix} u_L \\ d_L \end{pmatrix} + \bar{u}_R\sigma^\mu iD_\mu u_R + \bar{d}_R\sigma^\mu iD_\mu d_R && \text{(quark dynamical term)} \\
& -\frac{\sqrt{2}}{v} \left[(\bar{u}_L, \bar{d}_L)\phi M^d d_R + \bar{d}_R M^{d*}\bar{\phi} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \right] && \text{(down, strange, bottom mass term)} \\
& -\frac{\sqrt{2}}{v} \left[(-\bar{d}_L, \bar{u}_L)\phi^* M^u u_R + \bar{u}_R M^{u*}\phi^T \begin{pmatrix} -d_L \\ u_L \end{pmatrix} \right] && \text{(up, charmed, top mass term)} \\
& +\overline{(D_\mu\phi)}D^\mu\phi - m_h^2[\bar{\phi}\phi - v^2/2]^2/v^2, && \text{(Higgs dynamical and mass term)} \\
& +\text{(Hermitian conjugate of some terms)}. && \text{(1)}
\end{aligned}$$

where $\bar{\psi} = \psi^\dagger$, and the derivative operators are

$$D_\mu \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \left[\partial_\mu - \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu \right] \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad D_\mu \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \left[\partial_\mu + \frac{ig_1}{6}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu + ig\mathbf{G}_\mu \right] \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad (2)$$

$$D_\mu \nu_R = \partial_\mu \nu_R, \quad D_\mu e_R = [\partial_\mu - ig_1 B_\mu] e_R, \quad D_\mu u_R = \left[\partial_\mu + \frac{i2g_1}{3}B_\mu + ig\mathbf{G}_\mu \right] u_R, \quad D_\mu d_R = \left[\partial_\mu - \frac{ig_1}{3}B_\mu + ig\mathbf{G}_\mu \right] d_R, \quad (3)$$

$$D_\mu \phi = \left[\partial_\mu + \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu \right] \phi. \quad (4)$$

ϕ is a 2-component complex Higgs field. Since \mathcal{L} is $SU(2)$ gauge invariant, a gauge can be chosen so ϕ has the form

$$\phi^T = (0, v+h)/\sqrt{2}, \quad \langle \phi \rangle_0^T = (\text{expectation value of } \phi) = (0, v)/\sqrt{2}, \quad (5)$$

where v is a real constant such that $\mathcal{L}_\phi = \overline{(\partial_\mu\phi)}\partial^\mu\phi - m_h^2[\bar{\phi}\phi - v^2/2]^2/v^2$ is minimized, and h is a residual Higgs field. B_μ , \mathbf{W}_μ and \mathbf{G}_μ are the gauge boson vector potentials, and \mathbf{W}_μ and \mathbf{G}_μ are composed of 2×2 and 3×3 traceless Hermitian matrices. Their associated field tensors are

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad \mathbf{W}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + ig_2(\mathbf{W}_\mu \mathbf{W}_\nu - \mathbf{W}_\nu \mathbf{W}_\mu)/2, \quad \mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu + ig(\mathbf{G}_\mu \mathbf{G}_\nu - \mathbf{G}_\nu \mathbf{G}_\mu). \quad (6)$$

The non-matrix A_μ, Z_μ, W_μ^\pm bosons are mixtures of \mathbf{W}_μ and B_μ components, according to the weak mixing angle θ_w ,

$$A_\mu = W_{11\mu} \sin\theta_w + B_\mu \cos\theta_w, \quad Z_\mu = W_{11\mu} \cos\theta_w - B_\mu \sin\theta_w, \quad W_\mu^+ = W_\mu^{-*} = W_{12\mu}/\sqrt{2}, \quad (7)$$

$$B_\mu = A_\mu \cos\theta_w - Z_\mu \sin\theta_w, \quad W_{11\mu} = -W_{22\mu} = A_\mu \sin\theta_w + Z_\mu \cos\theta_w, \quad W_{12\mu} = W_{21\mu}^* = \sqrt{2}W_\mu^+, \quad \sin^2\theta_w = .2315(4). \quad (8)$$

The fermions include the leptons e_R, e_L, ν_R, ν_L and quarks u_R, u_L, d_R, d_L . They all have implicit 3-component generation indices, $e_i = (e, \mu, \tau)$, $\nu_i = (\nu_e, \nu_\mu, \nu_\tau)$, $u_i = (u, c, t)$, $d_i = (d, s, b)$, which contract into the fermion mass matrices $M_{ij}^e, M_{ij}^\nu, M_{ij}^u, M_{ij}^d$, and implicit 2-component indices which contract into the Pauli matrices,

$$\sigma^\mu = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \quad \tilde{\sigma}^\mu = [\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3], \quad \text{tr}(\sigma^i) = 0, \quad \sigma^{\mu\dagger} = \sigma^\mu, \quad \text{tr}(\sigma^\mu \sigma^\nu) = 2\delta^{\mu\nu}. \quad (9)$$

The quarks also have implicit 3-component color indices which contract into \mathbf{G}_μ . So \mathcal{L} really has implicit sums over 3-component generation indices, 2-component Pauli indices, 3-component color indices in the quark terms, and 2-component $SU(2)$ indices in $(\bar{\nu}_L, \bar{e}_L), (\bar{u}_L, \bar{d}_L), (-\bar{e}_L, \bar{\nu}_L), (-\bar{d}_L, \bar{u}_L), \bar{\phi}, \mathbf{W}_\mu, \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} -e_L \\ \nu_L \end{pmatrix}, \begin{pmatrix} -d_L \\ u_L \end{pmatrix}, \phi$.

The electroweak and strong coupling constants, Higgs vacuum expectation value (VEV), and Higgs mass are,

$$g_1 = e/\cos\theta_w, \quad g_2 = e/\sin\theta_w, \quad g = 3.892e, \quad v = \sqrt{2} \cdot 180 \text{ GeV} = 254 \text{ GeV}, \quad m_h \sim 115 - 180 \text{ GeV?} \quad (10)$$

where $e = \sqrt{4\pi\alpha\hbar c} = \sqrt{4\pi/137}$ in natural units. Using (4,5) and rewriting some things gives the mass of A_μ, Z_μ, W_μ^\pm ,

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{8}\text{tr}(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}) = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W_{\mu\nu}^-W^{+\mu\nu} + \left(\begin{array}{c} \text{higher} \\ \text{order terms} \end{array} \right), \quad (11)$$

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \quad W_{\mu\nu}^\pm = D_\mu W_\nu^\pm - D_\nu W_\mu^\pm, \quad D_\mu W_\nu^\pm = [\partial_\mu \pm ieA_\mu]W_\nu^\pm, \quad (12)$$

$$D_\mu \langle \phi \rangle_0 = \frac{iv}{\sqrt{2}} \begin{pmatrix} g_2 W_{12\mu}/2 \\ g_1 B_\mu/2 + g_2 W_{22\mu}/2 \end{pmatrix} = \frac{ig_2 v}{2} \begin{pmatrix} W_{12\mu}/\sqrt{2} \\ (B_\mu \sin\theta_w/\cos\theta_w + W_{22\mu})/\sqrt{2} \end{pmatrix} = \frac{ig_2 v}{2} \begin{pmatrix} W_\mu^+ \\ -Z_\mu/\sqrt{2} \cos\theta_w \end{pmatrix}, \quad (13)$$

$$\Rightarrow m_A = 0, \quad m_{W^\pm} = g_2 v/2 = 80.425(38) \text{ GeV}, \quad m_Z = g_2 v/2 \cos\theta_w = 91.1876(21) \text{ GeV}. \quad (14)$$

Ordinary 4-component Dirac fermions are composed of the left and right handed 2-component fields,

$$e = \begin{pmatrix} e_{L1} \\ e_{R1} \end{pmatrix}, \quad \nu_e = \begin{pmatrix} \nu_{L1} \\ \nu_{R1} \end{pmatrix}, \quad u = \begin{pmatrix} u_{L1} \\ u_{R1} \end{pmatrix}, \quad d = \begin{pmatrix} d_{L1} \\ d_{R1} \end{pmatrix}, \quad (\text{electron, electron neutrino, up and down quark}) \quad (15)$$

$$\mu = \begin{pmatrix} e_{L2} \\ e_{R2} \end{pmatrix}, \quad \nu_\mu = \begin{pmatrix} \nu_{L2} \\ \nu_{R2} \end{pmatrix}, \quad c = \begin{pmatrix} u_{L2} \\ u_{R2} \end{pmatrix}, \quad s = \begin{pmatrix} d_{L2} \\ d_{R2} \end{pmatrix}, \quad (\text{muon, muon neutrino, charmed and strange quark}) \quad (16)$$

$$\tau = \begin{pmatrix} e_{L3} \\ e_{R3} \end{pmatrix}, \quad \nu_\tau = \begin{pmatrix} \nu_{L3} \\ \nu_{R3} \end{pmatrix}, \quad t = \begin{pmatrix} u_{L3} \\ u_{R3} \end{pmatrix}, \quad b = \begin{pmatrix} d_{L3} \\ d_{R3} \end{pmatrix}, \quad (\text{tauon, tauon neutrino, top and bottom quark}) \quad (17)$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \quad \text{where } \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2I g^{\mu\nu}. \quad (\text{Dirac gamma matrices in chiral representation}) \quad (18)$$

The corresponding antiparticles are related to the particles according to $\psi^c = -i\gamma^2 \psi^*$ or $\psi_L^c = -i\sigma^2 \psi_R^*$, $\psi_R^c = i\sigma^2 \psi_L^*$. The fermion charges are the coefficients of A_μ when (8,10) are substituted into either the left or right handed derivative operators (2-4). The fermion masses are the singular values of the 3×3 fermion mass matrices M^ν, M^e, M^u, M^d ,

$$M^e = \mathbf{U}_L^{e\dagger} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \mathbf{U}_R^e, \quad M^\nu = \mathbf{U}_L^{\nu\dagger} \begin{pmatrix} m_{\nu_e} & 0 & 0 \\ 0 & m_{\nu_\mu} & 0 \\ 0 & 0 & m_{\nu_\tau} \end{pmatrix} \mathbf{U}_R^\nu, \quad M^u = \mathbf{U}_L^{u\dagger} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \mathbf{U}_R^u, \quad M^d = \mathbf{U}_L^{d\dagger} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \mathbf{U}_R^d, \quad (19)$$

$$m_e = .510998910(13) \text{ MeV}, \quad m_{\nu_e} \sim .001 - 2 \text{ eV}, \quad m_u = 1.5 - 3.3 \text{ MeV}, \quad m_d = 3.5 - 6 \text{ MeV}, \quad (20)$$

$$m_\mu = 105.658367(4) \text{ MeV}, \quad m_{\nu_\mu} \sim .001 - 2 \text{ eV}, \quad m_c = 1.16 - 1.34 \text{ GeV}, \quad m_s = 70 - 130 \text{ MeV}, \quad (21)$$

$$m_\tau = 1776.84(17) \text{ MeV}, \quad m_{\nu_\tau} \sim .001 - 2 \text{ eV}, \quad m_t = 169 - 174 \text{ GeV}, \quad m_b = 4.13 - 4.37 \text{ GeV}, \quad (22)$$

where the \mathbf{U} s are 3×3 unitary matrices ($\mathbf{U}^{-1} = \mathbf{U}^\dagger$). Consequently the ‘‘true fermions’’ with definite masses are actually linear combinations of those in \mathcal{L} , or conversely the fermions in \mathcal{L} are linear combinations of the true fermions,

$$e'_L = \mathbf{U}_L^e e_L, \quad e'_R = \mathbf{U}_R^e e_R, \quad \nu'_L = \mathbf{U}_L^\nu \nu_L, \quad \nu'_R = \mathbf{U}_R^\nu \nu_R, \quad u'_L = \mathbf{U}_L^u u_L, \quad u'_R = \mathbf{U}_R^u u_R, \quad d'_L = \mathbf{U}_L^d d_L, \quad d'_R = \mathbf{U}_R^d d_R, \quad (23)$$

$$e_L = \mathbf{U}_L^{e\dagger} e'_L, \quad e_R = \mathbf{U}_R^{e\dagger} e'_R, \quad \nu_L = \mathbf{U}_L^{\nu\dagger} \nu'_L, \quad \nu_R = \mathbf{U}_R^{\nu\dagger} \nu'_R, \quad u_L = \mathbf{U}_L^{u\dagger} u'_L, \quad u_R = \mathbf{U}_R^{u\dagger} u'_R, \quad d_L = \mathbf{U}_L^{d\dagger} d'_L, \quad d_R = \mathbf{U}_R^{d\dagger} d'_R. \quad (24)$$

When \mathcal{L} is written in terms of the true fermions, the \mathbf{U} s fall out except in $\bar{u}'_L \mathbf{U}_L^u \tilde{\sigma}^\mu W_\mu^\pm \mathbf{U}_L^{d\dagger} d'_L$ and $\bar{\nu}'_L \mathbf{U}_L^\nu \tilde{\sigma}^\mu W_\mu^\pm \mathbf{U}_L^{e\dagger} e'_L$. Because of this, and some absorption of constants into the fermion fields, the parameters in the \mathbf{U} s are entirely contained in only four components of the Cabibbo-Kobayashi-Maskawa matrix $\mathbf{V}^q = \mathbf{U}_L^q \mathbf{U}_L^{q\dagger}$ and four components of $\mathbf{V}^l = \mathbf{U}_L^l \mathbf{U}_L^{l\dagger}$. The unitary matrices \mathbf{V}^q and \mathbf{V}^l are often parameterized as

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta/2} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} e^{i\delta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta/2} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c_j = \sqrt{1 - s_j^2}, \quad (25)$$

$$\delta^q = 57(14) \text{ deg}, \quad s_{12}^q = 0.2243(16), \quad s_{23}^q = 0.0413(15), \quad s_{13}^q = 0.0037(5), \quad (26)$$

$$\delta^l = ?, \quad s_{12}^l = 0.57(3), \quad s_{23}^l = 0.7(1), \quad s_{13}^l = 0.0(2). \quad (27)$$

\mathcal{L} is invariant under a $U(1) \otimes SU(2)$ gauge transformation with $U^{-1} = U^\dagger$, $\det U = 1$, θ real,

$$\mathbf{W}_\mu \rightarrow U \mathbf{W}_\mu U^\dagger - (2i/g_2) U \partial_\mu U^\dagger, \quad \mathbf{W}_{\mu\nu} \rightarrow U \mathbf{W}_{\mu\nu} U^\dagger, \quad B_\mu \rightarrow B_\mu + (2/g_1) \partial_\mu \theta, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}, \quad \phi \rightarrow e^{-i\theta} U \phi, \quad (28)$$

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \rightarrow e^{i\theta} U \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix} \rightarrow e^{-i\theta/3} U \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \nu_R \rightarrow \nu_R, \quad u_R \rightarrow e^{-4i\theta/3} u_R, \quad e_R \rightarrow e^{2i\theta} e_R, \quad d_R \rightarrow e^{2i\theta/3} d_R, \quad (29)$$

and under an $SU(3)$ gauge transformation with $V^{-1} = V^\dagger$, $\det V = 1$,

$$\mathbf{G}_\mu \rightarrow V \mathbf{G}_\mu V^\dagger - (i/g) V \partial_\mu V^\dagger, \quad \mathbf{G}_{\mu\nu} \rightarrow V \mathbf{G}_{\mu\nu} V^\dagger, \quad u_L \rightarrow V u_L, \quad d_L \rightarrow V d_L, \quad u_R \rightarrow V u_R, \quad d_R \rightarrow V d_R. \quad (30)$$