The Georgi-Glashow Minimal SU(5) GUT Theory Lagrangian
From *Grand Unified Theories*, Graham G. Ross, 1985
and *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions*, Chris Quigg, 1983,
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Extracted by J.A. Shifflett, 20 Jan 2011.

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}\,tr(\mathbf{V}^{\mu\nu}\mathbf{V}_{\mu\nu}) + \bar{\psi} i\tilde{\sigma}^{\mu} D_{\mu}\psi + \bar{\Psi} i\tilde{\sigma}^{\mu} D_{\mu}\Psi && \text{(gauge term and fermion dyamical terms)}\\
& + \frac{\sqrt{2}}{v_5}\,(-\psi_j^T i\sigma^2 M^d \Psi^{jk}\phi_k + \epsilon_{jklmn}\Psi^{Tjk} i\sigma^2 M^u \Psi^{lm}\phi^n/4) && \text{(fermion mass terms)}\\
& + \overline{(D_\mu\phi)}D^\mu\phi - \frac{1}{2}\nu^2\bar{\phi}\phi + \frac{1}{4}\lambda(\bar{\phi}\phi)^2 + \alpha\bar{\phi}\phi\, tr(\Phi^2) + \beta\bar{\phi}\Phi^2\phi && \text{(Higgs } \mathbf{5} \text{ dynamical and mass terms)}\\
& + tr[\overline{(D_\mu\Phi)}D^\mu\Phi] - \mu^2 tr(\Phi^2) + \frac{a}{4}[tr(\Phi^2)]^2 + \frac{b}{2}tr(\Phi^4) && \text{(Higgs } \mathbf{24} \text{ dynamical and mass terms)}\\
& + \text{(Hermitian conjugate of some terms).} && (1)
\end{aligned}
$$

where $\bar{\psi} = \psi^{\dagger}$, and the derivative operators are

$$
D_\mu\psi = [\partial_\mu - ig_5\mathbf{V}_\mu]\,\psi, \quad D_\mu\Psi = [\partial_\mu + 2ig_5\mathbf{V}_\mu]\,\Psi, \quad D_\mu\phi = [\partial_\mu - ig_5\mathbf{V}_\mu]\phi, \quad D_\mu\Phi = \partial_\mu\Phi + ig_5(\Phi\mathbf{V}_\mu - \mathbf{V}_\mu\Phi). \tag{2}
$$

ϕ is a 5-component complex Higgs field and Φ is a 5×5 traceless real Higgs field. $\mathbf{V}_\mu$ is the vector potential composed of 5×5 traceless Hermitian matrices, with field tensor

$$
\mathbf{V}_{\mu\nu} = \partial_\mu\mathbf{V}_\nu - \partial_\nu\mathbf{V}_\mu + ig_2(\mathbf{V}_\mu\mathbf{V}_\nu - \mathbf{V}_\nu\mathbf{V}_\mu)/2. \tag{3}
$$

ψ is a 5-component complex fermion field and Ψ is a 5×5 antisymmetric fermion field. The Standard Model link is

$$
\mathbf{V}_\mu = \left(\begin{array}{c|cc} \mathbf{G}_\mu & \frac{X_\mu}{\sqrt{2}} & \frac{Y_\mu}{\sqrt{2}} \\ \hline X^\dagger_\mu/\sqrt{2} & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{W}_\mu/2$}} \\ Y^\dagger_\mu/\sqrt{2} & & \end{array}\right) + \sqrt{\frac{3}{5}}\,B_\mu \left(\begin{array}{c|c} -I/3 & 0 \\ \hline 0 & I/2 \end{array}\right), \quad \psi_i = \left(\begin{array}{c} d_i^c \\ \hline e \\ -\nu \end{array}\right), \quad \Psi^{ij} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|cc} \epsilon^{ijl}u_l^c & -u_i & -d_i \\ \hline u_j & 0 & -e^c \\ d_j & e^c & 0 \end{array}\right). \tag{4}
$$

$B_\mu, \mathbf{W}_\mu, \mathbf{G}_\mu$ are the usual gauge bosons and gluons, and $\mathbf{W}_\mu, \mathbf{G}_\mu$ are composed of 2×2 and 3×3 traceless Hermitian matrices. The new fields X_μ and Y_μ are called lepto-quark bosons, and they have implicit 3-component color indices. The fermions include the left-handed leptons and quarks e, ν, d_i, u_i, and their antiparticles e^c, ν^c, d_i^c, u_i^c, where i is a 3-component color index. The fermions all have implicit 3-component generation indices which contract into the fermion mass matrices M^u, M^d, and implicit 2-component indices which contract into the Pauli matrices,

$$
\sigma^\mu = \left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right], \quad \tilde{\sigma}^\mu = [\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3], \quad tr(\sigma^i) = 0, \quad \sigma^{\mu\dagger} = \sigma^\mu, \quad tr(\sigma^\mu\sigma^\nu) = 2\delta^{\mu\nu}. \tag{5}
$$

Using the identity $\epsilon^{ilm}\epsilon^{jln} = \delta^{ij}\delta^{mn} - \delta^{in}\delta^{mj}$ we have the outer products

$$
\bar{\psi}_i\psi_j = \left(\begin{array}{c|cc} \bar{d}_i^c d_j^c & \bar{d}_i^c e & -\bar{d}_i^c\nu \\ \hline \bar{e}d_j^c & \bar{e}e & -\bar{e}\nu \\ -\bar{\nu}d_j^c & -\bar{\nu}e & \bar{\nu}\nu \end{array}\right), \quad \bar{\Psi}^{il}\Psi^{jl} = \frac{1}{2}\left(\begin{array}{c|cc} \bar{u}_l^c u_l^c\delta_{ij} - \bar{u}_i^c u_j^c + \bar{u}_i u_j + \bar{d}_i d_j & -\epsilon^{iml}\bar{u}_l u_m & -\epsilon^{iml}\bar{u}_l d_m \\ \hline \epsilon^{jml}\bar{u}_l u_m & \bar{u}_l u_l + \bar{e}^c e^c & \bar{u}_l d_l \\ \epsilon^{jml}\bar{u}_l d_m & \bar{d}_l u_l & \bar{d}_l d_l + \bar{e}^c e^c \end{array}\right). \tag{6}
$$

Substituting (4,6) into (1) gives

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{8}tr(\mathbf{W}^{\mu\nu}\mathbf{W}_{\mu\nu}) - \frac{1}{2}tr(\mathbf{G}^{\mu\nu}\mathbf{G}_{\mu\nu}) + \text{(boson coupling terms involving } X_\mu \text{ and } Y_\mu)\\
& + (\bar{\nu},\bar{e})i\tilde{\sigma}^\mu\left(\partial_\mu - \frac{ig_5\sqrt{3/5}}{2}B_\mu + \frac{ig_5}{2}\mathbf{W}_\mu\right)\begin{pmatrix}\nu\\e\end{pmatrix} + (\bar{u},\bar{d})i\tilde{\sigma}^\mu\left(\partial_\mu + \frac{ig_5\sqrt{3/5}}{6}B_\mu + \frac{ig_5}{2}\mathbf{W}_\mu + ig_5\mathbf{G}_\mu\right)\begin{pmatrix}u\\d\end{pmatrix}\\
& + \bar{e}^c i\tilde{\sigma}^\mu\left(\partial_\mu + ig_5\sqrt{3/5}\,B_\mu\right)e^c + \bar{u}^c i\tilde{\sigma}^\mu\left(\partial_\mu - \frac{2ig_5\sqrt{3/5}}{3}B_\mu - ig_5\mathbf{G}_\mu\right)u^c + \bar{d}^c i\tilde{\sigma}^\mu\left(\partial_\mu + \frac{ig_5\sqrt{3/5}}{3}B_\mu - ig_5\mathbf{G}_\mu\right)d^c\\
& + \frac{g_5}{\sqrt{2}}\,(\bar{\nu}\tilde{\sigma}^\mu Y_\mu d^c - \bar{e}\tilde{\sigma}^\mu X_\mu d^c + \bar{d}^c\tilde{\sigma}^\mu Y_\mu^\dagger\nu - \bar{d}^c\tilde{\sigma}^\mu X_\mu^\dagger e) + \frac{g_5}{\sqrt{2}}\,\epsilon^{iml}(\bar{u}_l\tilde{\sigma}^\mu X_{i\mu}u_m + \bar{u}_l\tilde{\sigma}^\mu Y_{i\mu}d_m - \bar{u}_l\tilde{\sigma}^\mu X_{i\mu}^\dagger u_m - \bar{u}_l\tilde{\sigma}^\mu Y_{i\mu}^\dagger d_m)\\
& + \text{(fermion mass terms and Higgs terms),} && (7)
\end{aligned}
$$

where we are using the usual definitions of the Standard Model field tensors,

$$
B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad \mathbf{W}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu + ig_2(\mathbf{W}_\mu\mathbf{W}_\nu - \mathbf{W}_\nu\mathbf{W}_\mu)/2, \quad \mathbf{G}_{\mu\nu} = \partial_\mu\mathbf{G}_\nu - \partial_\nu\mathbf{G}_\mu + ig(\mathbf{G}_\mu\mathbf{G}_\nu - \mathbf{G}_\nu\mathbf{G}_\mu). \tag{8}
$$

Unlike the Standard Model, right-handed fields are mostly represented in the Lagrangian (1) by the antiparticles according to $\psi^c = -i\sigma^2\psi_R^*,\ \psi_R^c = i\sigma^2\psi^*$. Using this definition and the identities (5) and $\sigma^2\tilde{\sigma}^\mu\sigma^2 = \sigma^{\mu T}$ gives

$$\bar{\psi}^c\tilde{\sigma}^\mu\psi^c \;=\; (-i\sigma^2\psi_R^*)^\dagger\tilde{\sigma}^\mu(-i\sigma^2\psi_R^*) = \psi_R^T\sigma^2\tilde{\sigma}^\mu\sigma^2\psi_R^* = \psi_R^T\sigma^{\mu T}\psi_R^* = -\psi_R^\dagger\sigma^\mu\psi_R \tag{9}$$

$$\bar{\psi}^c\tilde{\sigma}^\mu\partial_\mu\psi^c \;=\; -\partial_\mu\psi_R^\dagger\sigma^\mu\psi_R = \psi_R^\dagger\sigma^\mu\partial_\mu\psi_R + (\mathit{total\ derivative}). \tag{10}$$

The sign change in the last step of (9) is because fermions anticommute, and (10) comes from (9) and integration by parts. Comparing (7) with the Standard Model using (9,10) we find that at high energies the strong and electroweak coupling constants are equal, and the weak mixing angle is a bit off from the measured value of $sin^2\theta_w = .23$,

$$g_5 = g = g_2 = e/sin\theta_w, \quad g_5\sqrt{3/5} = g_1 = e/cos\theta_w \quad \Rightarrow \ sin^2\theta_w = g_1^2/(g_1^2+g_2^2) = 3/8. \tag{11}$$

However, low energy values after quantum corrections come out fairly close to measurement. The Higgs fields are assumed to take on vacuum expectation values (VEVs) of the form

$$<\!\phi\!>_0^\dagger = \frac{v_5}{\sqrt{2}}(\,0,0,0 \mid 0,1\,), \qquad <\!\Phi\!>_0 = v_{24}\begin{pmatrix} I & \mid & 0 \\ --- & \mid & --- \\ 0 & \mid & -3I/2 \end{pmatrix}, \qquad \text{where} \ \ v_5 = 246GeV, \ \ v_{24} \sim 10^{15}GeV. \tag{12}$$

The Higgs **24** derivative operator in (1,2) gives masses to the lepto-quark bosons X_μ and Y_μ, but not to $\mathbf{G}_\mu$ or $\mathbf{W}_\mu$,

$$D_\mu\!<\!\Phi\!>_0 = ig_5(<\!\Phi\!>_0\mathbf{V}_\mu - \mathbf{V}_\mu<\!\Phi\!>_0) = \frac{5iv_{24}g_5}{2}\begin{pmatrix} 0 & \mid & \frac{X_\mu}{\sqrt{2}} \ \ \frac{Y_\mu}{\sqrt{2}} \\ --- & \mid & ----- \\ -X_\mu^\dagger/\sqrt{2} & & \\ -Y_\mu^\dagger/\sqrt{2} & \mid & 0 \end{pmatrix} \ \Rightarrow m_X = m_Y = \frac{5v_{24}g_5}{2\sqrt{2}} \sim 10^{15}GeV. \tag{13}$$

The Higgs **5** derivative operator in (1,2) gives correct masses to the Standard Model $W_\mu^\pm$ and Z_μ gauge bosons,

$$D_\mu\!<\!\phi\!>_0 = -ig_5V_\mu<\!\phi\!>_0 = -\frac{iv_5g_5}{\sqrt{2}}\begin{pmatrix} -\frac{Y_\mu}{\sqrt{2}} \\ -------- \\ W_{12\mu}/2 \\ W_{22\mu}/2+\sqrt{3/5}\,B_\mu/2 \end{pmatrix} = \frac{iv_5g_5}{2}\begin{pmatrix} Y_\mu \\ ---- \\ -W_\mu^+ \\ Z_\mu/\sqrt{2}\cos\theta_w \end{pmatrix} \quad \Rightarrow m_{W^\pm} = m_Z\cos\theta_w = \frac{v_5g_5}{2}, \tag{14}$$

$$\text{where} \quad W_\mu^+ = W_{12\mu}/\sqrt{2}, \quad W_\mu^- = W_\mu^{+*}, \quad Z_\mu = -(W_{22\mu}\cos\theta_w + \sin\theta_w B_\mu), \quad \sqrt{3/5} = g_1/g_5 = \sin\theta_w/\cos\theta_w. \tag{15}$$

It also adds an insignificant mass contribution to Y_μ. The ϕ, Φ coupling in the Higgs **5** term avoids massless Higgs fields and renormalization problems. Substituting (12) into the fermion mass term in (1) gives

$$\mathcal{L}_m = -\psi_j^T i\sigma^2 M^d\Psi^{j5} + \epsilon_{jklm5}\Psi^{Tjk}i\sigma^2M^u\Psi^{lm}/4 + h.c. \;=\; -\bar{d}_RM^dd - \bar{e}_R^cM^de^c - \bar{u}_RM^uu/2 - \bar{u}_R^cM^uu^c/2 + h.c. \tag{16}$$

$$= -\bar{d}_RM^dd - \bar{e}_RM^{dT}e - \bar{u}_RM^uu/2 - \bar{u}_RM^{uT}u/2 + h.c. \tag{17}$$

where $h.c.$ means Hermitian conjugate. If we assume (with no justification) that the fermion mass matrices M^d, M^u are symmetric then things are much like the Standard Model. The fermion masses are the singular values of M^d, M^u,

$$M^d = \mathbf{U}_L^{d\dagger}\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}\mathbf{U}_R^d, \quad M^u = \mathbf{U}_L^{u\dagger}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}\mathbf{U}_R^u, \tag{18}$$

where the $\mathbf{U}$s are 3×3 unitary matrices ($\mathbf{U}^{-1} = \mathbf{U}^\dagger$). Consequently the "true fermions" with definite masses are actually linear combinations of those in $\mathcal{L}$, or conversely the fermions in $\mathcal{L}$ are linear combinations of the true fermions,

$$d'_L = \mathbf{U}_L^dd_L, \quad d'_R = \mathbf{U}_R^dd_R, \quad e'_L = \mathbf{U}_L^de_L, \quad e'_R = \mathbf{U}_R^de_R, \quad u'_L = \mathbf{U}_L^uu_L, \quad u'_R = \mathbf{U}_R^uu_R, \tag{19}$$

$$d_L = \mathbf{U}_L^{d\dagger}d'_L, \quad d_R = \mathbf{U}_R^{d\dagger}d'_R, \quad e_L = \mathbf{U}_L^{d\dagger}e'_L, \quad e_R = \mathbf{U}_R^{d\dagger}e'_R, \quad u_L = \mathbf{U}_L^{u\dagger}u'_L, \quad u_R = \mathbf{U}_R^{u\dagger}u'_R. \tag{20}$$

When $\mathcal{L}$ is written in terms of the true fermions, the $\mathbf{U}$s fall out except in one term containing the unitary matrix $V = \mathbf{U}_L^u\mathbf{U}_L^{d\dagger}$, which is analogous to the Cabibbo-Kobayashi-Maskawa matrix in the Standard Model. From (17,18) we see that the high energy mass of the electron is the same as the down quark, and likewise for the other generations,

$$m_e = m_d, \qquad m_\mu = m_s, \qquad m_\tau = m_b. \tag{21}$$

Low energy predictions after quantum corrections are closer to measurement, but one mass prediction is still off by a factor of 8. The proton decay time from the lepton-quark interaction terms in (7) also disagrees with measurement,

$$\Gamma^{-1}_{\text{predicted}}(p \to e^+\pi^o) = 4.5 \times 10^{29\pm1.7}\text{years}, \qquad \Gamma^{-1}_{\text{measured}}(p \to e^+\pi^o) > 6 \times 10^{31}\text{years}. \tag{22}$$

$\mathcal{L}$ is invariant under a $SU(5)$ gauge transformation with $U^{-1} = U^\dagger,\ detU = 1$,

$$\mathbf{V}_\mu \to U\mathbf{V}_\mu U^\dagger - (i/g_5)U\partial_\mu U^\dagger, \quad \mathbf{V}_{\mu\nu} \to U\mathbf{V}_{\mu\nu}U^\dagger, \quad \psi \to U\psi, \quad \Psi \to U\Psi U^\dagger, \quad \phi \to U\phi, \quad \Phi \to U\Phi U^\dagger. \tag{23}$$