Mercury Temperature vs. Latitude, Depth and Time

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Here we find an equation for the temperature on Mercury vs. latitude, depth and time. We also find the depth underground required for a given daily temperature fluctuation. And we calculate the range of latitude and depth where there is room temperature, $T\!=\!295\!\pm\!1K$.

From energy conservation and Fourier's law $(heat\ flux)\!=\!-k\nabla T$, we get the one dimensional heat diffusion equation,

$$\frac{dT}{dt} = \frac{1}{\rho c}\frac{d}{dz}\left(k\frac{dT}{dz}\right) \tag{1}$$

where

$$\begin{aligned} k &= (heat\ conductivity), &(2)\\ \rho c &= (density)\times(heat\ capacity) = 10^6 J/m^3K, &(3)\\ T &= (temperature), &(4)\\ z &= (depth), &(5)\\ t &= (time). &(6)\end{aligned}$$

The heat conductivity is assumed to increase from that of lunar dust in the first 2cm, to lunar regolith within 2cm-5m, to fractured rock at 5m underground. The analysis is much simplified if we assume that $\sqrt{k}$ depends linearly on depth within this region, which is where the analysis applies. Data from several sources suggests the following expression,

$$\sqrt{k} = \sqrt{.001} + \sqrt{.02}\,z\ \sqrt{W/mK}. \tag{7}$$

To solve the heat diffusion equation (1) let us rewrite it as follows

$$\frac{dT}{dt} = \frac{\pi}{\tau}\frac{d}{dz}\left(D^2\frac{dT}{dz}\right), \tag{8}$$

where

$$\begin{aligned} D &= \sqrt{\tau k/\pi\rho c} = D_0 + \beta z, &(9)\\ D_0 &= \sqrt{\tau k_0/\pi\rho c} = .07m, &(10)\\ k_0 &= (surface\ heat\ conductivity) = .001W/mK, &(11)\\ \beta &= (dimensionless\ heat\ conductivity\ gradient\ parameter) = .3, &(12)\\ \tau &= (length\ of\ day) = 176\ Earth\ days = 1.52\times 10^7 secs. &(13)\end{aligned}$$

Using the chain rule and (9), the heat diffusion equation (8) becomes

$$\frac{\tau}{\pi}\frac{dT}{dt} = \frac{dD}{dz}D\frac{dT}{dz} + D\frac{d}{dz}\left(D\frac{dT}{dz}\right) = \beta\frac{dT}{d\zeta} + \frac{d^2T}{d\zeta^2}, \tag{14}$$

where we changed depth variables from z to ζ with

$$d\zeta = \frac{dz}{D}, \qquad \zeta = \int\frac{dz}{D_0+\beta z} = \frac{ln(D_0+\beta z)}{\beta} - \frac{ln(D_0)}{\beta} = \frac{1}{\beta}ln\left(1+\frac{\beta z}{D_0}\right), \tag{15}$$

$$z = \frac{D_0}{\beta}(e^{\beta\zeta}-1) = D_0\zeta\left(1+\frac{\beta\zeta}{2!}+\frac{(\beta\zeta)^2}{3!}\ldots\right). \tag{16}$$

Assuming a sinusoidal time dependence (which would be roughly true), equation (14) has the solution

$$T = \bar{T} + \widehat{T}Re[e^{-\alpha\zeta-i2\pi t/\tau}] = \bar{T} + \widehat{T}e^{-\alpha'\zeta}cos(\alpha''\zeta + 2\pi t/\tau), \qquad \alpha = \alpha' + i\alpha'', \tag{17}$$

where insertion into the differential equation gives

$$-2i = -\beta\alpha + \alpha^2, \tag{18}$$

$$\alpha = \frac{\beta+\sqrt{\beta^2-8i}}{2} = \frac{\beta}{2} + \frac{\sqrt{\sqrt{64+\beta^4}+\beta^2}}{2\sqrt{2}} - \frac{i\sqrt{\sqrt{64+\beta^4}-\beta^2}}{2\sqrt{2}} = (1-i)\left(1+\frac{i\beta^2}{16}\ldots\right)+\frac{\beta}{2} = 1.16-.99i, \tag{19}$$

$$\bar{T} = \frac{1}{2}(T_{max}+T_{min}), \quad \widehat{T} = \frac{1}{2}(T_{max}-T_{min}). \tag{20}$$

The conducted power flux traveling into the surface is found using (15) and the derivative of (17) at z=0,

$$F_{conducted}(t) = -\left[k\frac{dT}{dz}\right]_{z=0} = -k_0\left[\frac{d\zeta}{dz}\frac{dT}{d\zeta}\right]_{z=0} = \frac{k_0\widehat{T}}{D_0}\left[\alpha' e^{-\alpha'\zeta}cos(\alpha''\zeta + 2\pi t/\tau) + e^{-\alpha'\zeta}\alpha'' sin(\alpha''\zeta + 2\pi t/\tau)\right]_{\zeta=0} \quad (21)$$

$$= \frac{k_0\widehat{T}}{D_0}\left[\alpha' cos(2\pi t/\tau) + \alpha'' sin(2\pi t/\tau)\right], \quad (22)$$

$$F_{conducted}(0) = \frac{\alpha' k_0\widehat{T}}{D_0}, \qquad F_{conducted}(\tau/2) = -\frac{\alpha' k_0\widehat{T}}{D_0}. \quad (23)$$

The emitted power flux is found from the Stephan-Boltzmann law,

$$F_{emitted} = \sigma T^4 \quad (24)$$

where

$$\sigma = (Stephan - Boltzmann\ constant) = 5.67 \times 10^{-8} W/m^2K^4. \quad (25)$$

To calculate T_{max} we equate the absorbed minus emitted power flux to the conducted power flux at noon ($t = 0$),

$$(1-A)cos(\phi)\sigma T_{Sun}^4\left(\frac{R_{Sun}}{R_{orbit}}\right)^2 - \sigma T_{max}^4 = \frac{\alpha' k_0\widehat{T}}{D_0}, \quad (26)$$

where

$$A = (albedo) = .06, \quad (27)$$

$$\phi = (latitude), \quad (28)$$

$$T_{Sun} = (Sun\ surface\ temperature) = 5785K, \quad (29)$$

$$R_{Sun} = (Sun\ radius) = .00465AU, \quad (30)$$

$$R_{orbit} = (Mercury\ orbital\ radius) = .31AU\ at\ perihelion,\ .47AU\ at\ aphelion. \quad (31)$$

The conducted power flux is far below the absorbed and emitted power flux (as shown below). Neglecting it gives

$$T_{max} = ((1-A)cos(\phi))^{1/4}T_{Sun}\sqrt{\frac{R_{Sun}}{R_{orbit}}}. \quad (32)$$

Calculating the max temperatures at perihelion and aphelion gives,

$$T_{maxp} = 698K cos^{1/4}(\phi), \qquad T_{maxa} = 567K cos^{1/4}(\phi). \quad (33)$$

At ϕ=0 these formulas are close to the measured maximum equatorial temperatures of 700K and 600K.

To calculate T_{min} we equate the emitted power flux to the conducted power flux at midnight ($t = \tau/2$)

$$\sigma T_{min}^4 = \frac{\alpha' k_0\widehat{T}}{D_0} = \frac{\alpha' k_0(T_{max} - T_{min})}{2D_0} = \frac{\alpha' k_0 T_{max}}{2D_0}\left(1 - \frac{T_{min}}{T_{max}}\right). \quad (34)$$

Using the approximation $(1+x)^{1/n} \approx 1 + x/n$ for $x \ll 1$ gives

$$T_{min} = \left(\frac{\alpha' k_0 T_{max}}{2D_0\sigma}\right)^{1/4}\left(1 - \frac{T_{min}}{4T_{max}}\right), \quad (35)$$

$$T_{min} = \left[\left(\frac{2D_0\sigma}{\alpha' k_0 T_{max}}\right)^{1/4} + \frac{1}{4T_{max}}\right]^{-1}, \qquad for \quad T_{max} > \left(\frac{3}{4}\right)^{4/3}\left(\frac{\alpha' k_0}{2D_0\sigma}\right)^{1/3} = 36K. \quad (36)$$

Substituting (33) into (36) using (11,10,12,19) gives a minimum temperature at perihelion and aphelion

$$T_{minp} = \left[\frac{cos^{-1/16}(\phi)}{100.5K} + \frac{cos^{-1/4}(\phi)}{2792K}\right]^{-1}, \qquad T_{mina} = \left[\frac{cos^{-1/16}(\phi)}{95.4K} + \frac{cos^{-1/4}(\phi)}{2268K}\right]^{-1}. \quad (37)$$

At ϕ=0 we get T_{minp}=97K, T_{mina}=92K which are close to the measured minimum equatorial temperature of 90K.

The orbit of Mercury is synchronized with its rotation such that 0° and 180° longitudes experience midnight and noon at perihelion whereas 90° and 270° longitudes experience midnight and noon at aphelion. At perihelion and $\phi = \pm 74.5°$ latitude we have

$$T_{maxp} = 502K, \quad T_{minp} = 88K, \quad \bar{T}_p = 295K, \quad \widehat{T}_p = 207K, \tag{38}$$

$$\frac{d\bar{T}_p}{d\phi} = -\frac{T_{maxp}\,tan(\phi)}{8} - \frac{T_{minp}\,tan(\phi)}{8}\left[1 - \frac{3T_{minp}\,cos^{-1/16}(\phi)}{4\times 100.5K}\right] = -4.15K/°latitude, \qquad \frac{d\bar{T}_p}{dl} = .097K/km. \tag{39}$$

At aphelion and $\phi = \pm 52.2°$ latitude we have

$$T_{maxa} = 502K, \quad T_{mina} = 88K, \quad \bar{T}_a = 295K, \quad \widehat{T}_a = 207K, \tag{40}$$

$$\frac{d\bar{T}_a}{d\phi} = -\frac{T_{maxa}\,tan(\phi)}{8} - \frac{T_{mina}\,tan(\phi)}{8}\left[1 - \frac{3T_{mina}\,cos^{-1/16}(\phi)}{4\times 95.4K}\right] = -1.48K/°latitude, \qquad \frac{d\bar{T}_a}{dl} = .035K/km. \tag{41}$$

Here we have used

$$R_{Mercury} = (radius\ of\ Mercury) = 2.44\times 10^3 km \quad \Rightarrow \quad dl/d\phi = -2.44\times 10^3\times\pi/180 = -42.586\,km/°latitude. \tag{42}$$

So there are two rings circling Mercury where $\bar{T} = 295\pm 1K$, one in each hemisphere. They pass through $\pm 74.5°$ latitude (with .48°, 21km width) at 0° and 180° longitude, and dip down to $\pm 52.2°$ latitude (with 1.36°, 58km width) at 90° and 270° longitude. Using (16,17) gives the daily temperature fluctuation T_Δ as a function of depth,

$$T_\Delta = \widehat{T}e^{-\alpha'\zeta} = \widehat{T}\left(1 + \frac{\beta z}{D_0}\right)^{-\alpha'/\beta}. \tag{43}$$

Inverting this gives the distance one has to go underground to get T_Δ of temperature fluctuation,

$$z = \frac{D_0}{\beta}\left((T_\Delta/\widehat{T})^{-\beta/\alpha'} - 1\right). \tag{44}$$

At the latitudes mentioned above, using (10,12,19,38) and $T_\Delta = 1K$ we find that the temperature is nearly constant at room temperature less than a meter below the surface,

$$z = .7m \quad \Rightarrow \quad T = 295\pm 1K. \tag{45}$$

Finally, let us discuss some approximations used above. First, when calculating T_{max} we assumed that $(conducted\ power\ flux) = \alpha_0 k_0 \widehat{T}/D_0$ was small compared to $(emitted\ power\ flux) = \sigma T_{max}^4$. Calculating the ratio at aphelion gives $(conducted\ power\ flux)/(emitted\ power\ flux) \sim 10^{-3}$, which justifies the approximation. Another implicit approximation is that $(equator\ to\ pole\ heat\ flux) \sim \alpha' k_0 T_{max}/R_{Mercury}$ is negligible compared to $(emitted\ power\ flux) = \sigma T_{min}^4$. Calculating the ratio using T_{max} at $\phi = 0°$ and T_{min} at $\phi = 74.5°$ gives $(equator\ to\ pole\ heat\ flux)/(emitted\ power\ flux) \sim 10^{-7}$, which justifies this approximation. Also, throughout this analysis we have been assuming that the conductivity is independent of temperature, and that the temperature has a sinusoidal time dependence. In reality the conductivity of the material very close to the surface probably has a temperature-dependent radiative component, perhaps with $k = [1 + (T/315K)^3]\times .001\,W/mK$ according to one source. In reality the solar light intensity driving function goes as $max[0, cos(2\pi t/\tau)]$ instead of $[1 + cos(2\pi t/\tau)]/2$, so there will be higher order modes. Our neglect of the conductivity temperature dependence and higher order modes certainly introduces error into the analysis, but these approximations are at least partially justified by the agreement of our T_{max} and T_{min} values with measurement.

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